

NUMERICAL SOLUTION OF STEFAN PROBLEMS BY THE METHOD OF GREEN FUNCTIONS

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A method for reducing a multidimensional Stefan problem to a system of Hammerstein integral equations is proposed. Application of the proposed method to numerical solution of one-dimensional nonstationary Stefan problems formulated for the cases of an internal phase front, coincidence of the phase front with the external boundary, and a movable external boundary is considered. The efficiency of the method is tested on an exactly solvable Stefan problem.

It is well known that the majority of numerical methods of solving heat-conduction equations with a phase transition (Stefan problems) can be divided into two groups: direct-computation schemes [1, 2] and schemes based on explicit isolation of the front [3-6]. Efficient difference algorithms for direct computation are especially widely applied to multidimensional problems. However, the accuracy of computing both the temperature and the position of the phase interface depend strongly on the smoothing parameter, whose evaluation is rather complicated. In implementing difference schemes based on explicit isolation of the front, considerable difficulties arise due to their complex character and insufficient efficiency resulting from the need to adjust the grid in each time step. At the same time, the possibilities of application of analytical methods at the initial stage of the investigation of problems of this type have by no means been exhausted, in particular, use of the method of Green functions (here, the Green function of the corresponding linear problem for the Laplace operator is meant) to reduce Stefan problems to integral equations. The advantage of this approach as compared to finite-difference methods consists in that, first, no approximation of the sought solution over spatial variables is necessary and, second, more efficient projection-grid methods [7] can be used to solve the integral equations obtained. The efficiency of computations can also be improved by using a stable computational grid that does not need to be adjusted in each time step.

In what follows, we consider a general formulation of the Stefan problem with account for the temperature dependence of the coefficients of the heat-conduction equation (intrinsic nonlinearity) under nonlinear boundary conditions (extrinsic nonlinearity). Application of the Kirchhoff transform linearizes the elliptic part of the nonlinear heat-conduction equation. This, however, even in the simplest case of linear boundary conditions of the third kind (describing heat transfer obeying Newton's law), transforms them into nonlinear ones. Subsequent use of the method of Green functions [8, 9] becomes impossible (excluding the case of boundary conditions of the first kind), since the corresponding Green function cannot exist as a Green function for the second internal boundary-value problem for the Laplace operator [8]. We propose a method for reducing the Kirchhoff-transformed Stefan problem to integral equations (in the case of a nonstationary problem the method of lines is first applied).

To explain the essence of the method proposed, let us consider a multidimensional stationary Stefan problem. The method is applicable to an arbitrary number of phases; however, to simplify considerations, we restrict ourselves to the two-phase problem. Let the body being heated occupy a domain Ω bounded by a closed piecewise-smooth surface S . The phase-transition surface Γ , whose position is unknown, divides the domain Ω into the two subdomains $\Omega_s = \{P \in \Omega: T(P) < T_*\}$ and $\Omega_L = \{P \in \Omega: T(P) > T_*\}$ occupied by the solid and liquid phases of the substance, respectively. In each of the domains Ω_s and Ω_L the temperature $T(P)$ at the point P satisfies the stationary heat conduction equation:

$$\operatorname{div}(\lambda_s(T(P)) \operatorname{grad} T(P)) = 0, \quad P \in \Omega_s, \quad \operatorname{div}(\lambda_L(T(P)) \operatorname{grad} T(P)) = 0, \quad P \in \Omega_L,$$

where $\lambda_s(T)$ and $\lambda_l(T)$ are given continuously differentiable functions. A nonlinear boundary condition determined by the regularities of the heat transfer is specified on the surface S :

$$\lambda(T(P)) \frac{\partial T(P)}{\partial n} = q(P, T(P)), \quad P \in S, \quad \lambda(T) = \begin{cases} \lambda_s(T), & T < T_*, \\ \lambda_l(T), & T \geq T_*, \end{cases}$$

where $\partial T / \partial n$ is the derivative along the external normal to S . On the sought phase interface Γ , in addition to the fact that the temperature $T(P)$ equals the temperature of the phase transition $T(P) = T_*$, $P \in \Gamma$, the stationary Stefan condition is satisfied:

$$\lambda_s(T(P)) \frac{\partial T(P)}{\partial n} \Big|_{\substack{P \in \Omega_s \\ P \rightarrow \Gamma}} = \lambda_l(T(P)) \frac{\partial T(P)}{\partial n} \Big|_{\substack{P \in \Omega_l \\ P \rightarrow \Gamma}}. \quad (1)$$

By applying the Kirchhoff transform

$$u(P) = u(T) = \int_{T_0}^{T(P)} \lambda(T) dT, \quad T_0 \leq \min_{P \in \bar{\Omega}} T(P)$$

(the inverse transform exists, for example, when the dependences $\lambda_s(T)$ and $\lambda_l(T)$ are linear) and taking into account the condition of equality of heat fluxes (1), owing to which the function $u(P)$ is continuously differentiable even in passing through the surface Γ , we reduce the original problem to a nonlinear boundary-value problem for the Laplace equation:

$$\Delta u(P) = 0, \quad P \in \Omega, \quad \frac{\partial u(P)}{\partial n} = f(P, u(P)), \quad P \in S \quad (2)$$

(here $f(P, u(P)) = q(P, T(u(P)))$) and an additional condition for determination of the surface Γ :

$$u(P) = u_*, \quad P \in \Gamma. \quad (3)$$

If one tries to reduce problem (2) to an integral equation using the method of Green functions, the following boundary-value problem is obtained for determination of the Green function $G(P, Q)$:

$$\Delta G(P, Q) = -\delta(P, Q), \quad P \in \Omega, \quad \frac{\partial G(P, Q)}{\partial n} = 0, \quad P \in S$$

(here $\delta(P, Q)$ is Dirac's δ -function with a singularity at the point $Q \in \Omega$), which has no solution [8].

However, the problem arising can be solved by constructing a Green function that we will refer to as auxiliary. Let $S = S_1 \cup S_2$, where S_1 is a smooth surface and S_2 is a smooth or piecewise smooth surface. Let us write the boundary condition of problem (2) in the form

$$\frac{\partial u(P)}{\partial n} + hu(P) = f(P, u(P)) + hu(P), \quad P \in S_1, \quad \frac{\partial u(P)}{\partial n} = f(P, u(P)), \quad P \in S_2.$$

where h is an arbitrary positive number ($0 < h < \infty$). Now we can introduce an auxiliary Green function as the solution of the following boundary-value problem:

$$\Delta G(P, Q) = -\delta(P, Q), \quad P \in \Omega, \quad \frac{\partial G(P, Q)}{\partial n} + hG = 0, \quad P \in S_1, \quad \frac{\partial G(P, Q)}{\partial n} = 0, \quad P \in S_2. \quad (4)$$

This function exists and is unique [8], and the possibility of determining it in practice depends on the shape of the surface S . It should be noted that the boundary condition of the third kind with the auxiliary parameter h introduced can be considered on the entire surface S , not just on its smooth portion S_1 .

If the Green function is found from (4), then with its help, by using the second Green formula for the Laplace operator applied to the functions u and G , one can reduce boundary-value problem (2) to the integral relationship

$$u(P) = \int_{S_1} G(P, Q) [f(Q, u(Q)) + hu(Q)] dS_Q + \int_{S_2} G(P, Q) f(Q, u(Q)) dS_Q, \quad P \in \Omega. \quad (5)$$

By considering this relationship on each of the smooth surfaces S_1 and S_{2i} , $i = \overline{1, n}$ ($\bigcup_{i=1}^n S_{2i} = S_2$), we obtain a system of Hammerstein integral equations with respect to values of the function $u(P)$ on each of these surfaces. After numerical solution of it, the function $u(P)$ is determined by relationship (5) uniquely (i.e., it does not depend on the value of the parameter h), since Eq. (5) was obtained after an equivalent transformation of the boundary conditions of problem (2). The phase-interface surface is determined numerically from Eq. (3).

We applied the auxiliary Green function method to numerical solution of an axisymmetric stationary Stefan problem arising in mathematical modeling of the steady-state temperature regime in the cathode-ray autocrucible melting [10]. In this case the domain $\Omega = \{(r, z): 0 < z < a; 0 < r < l\}$ is a cylinder of radius a and height l . A linear temperature dependence of the thermal conductivity of the solid phase was considered. Numerical solution of the system of Hammerstein integral equations was carried out using a projection-grid method [7]. The results obtained were independent of the value of the auxiliary parameter h .

Let us consider the application of the auxiliary Green function method to solution of one-dimensional time-dependent Stefan problems, again restricting considerations to the case of two phases, although the method is applicable to an arbitrary number of phases. We will consider only boundary conditions that require construction of an auxiliary Green function, thereby including cases of both nonlinear boundary conditions and linear boundary conditions of the second and third kind. Stefan problems with boundary conditions of the first kind (when a temperature variation law is specified on a portion of the boundary) are reduced to a system of integral equations with the use of an ordinary Green function.

First we consider problems with an internal phase front $x = z(t)$ assuming for definiteness that $\dot{z}(t) > 0$:

$$\begin{aligned} L_k T &= -w(x, t, T) + pz(t) x^k \delta_{(k)}(x - z(t)), \quad 0 < x < R, \quad t > 0; \quad k = 0; 1; 2; \\ T(x, 0) &= T^0(x); \quad T^0(z_0) = T_*; \quad T(z(t), t) = T_*; \quad z(0) = z_0, \quad 0 \leq z_0 < R; \\ \lambda(T) T_x &= -\varepsilon_k q_0(t, T), \quad x = 0; \quad \lambda(T) T_x = q_R(t, T), \quad x = R. \end{aligned} \quad (6)$$

Here the values $k = 0, 1, 2$ correspond to cases of plane, cylindrical, and spherical symmetry of the temperature field $T(x, t)$, respectively, $\varepsilon_0 = 1$, $\varepsilon_1 = \varepsilon_2 = 0$, and $L_k T = x^{-k} \partial(\lambda(T) x^k T_x) / \partial x - \gamma(T) T_t$. The specified functions $\lambda(T)$, $\gamma(T)$, and $w(x, t, T)$ can be discontinuous upon crossing the phase interface.

By applying the Kirchoff transform $u(T)$ to the function $T(x, t)$ and to its values at $t = 0$ and $x = z(t)$: $u(T^0(x)) = u_0(x)$ and $u(T_*) = u_*$, we obtain a Stefan problem for evaluation of the functions $u(x, t)$ and $z(t)$, in a form convenient for subsequent solution:

$$\begin{aligned} \Delta_k u &= \kappa(u) u_t - W(x, t, u) + pz(t) x^k \delta_{(k)}(x - z(t)), \quad 0 < x < R, \quad t > 0; \\ u(x, 0) &= u_0(x); \quad u_0(z_0) = u_*; \quad u(z(t), t) = u_*; \quad z(0) = z_0; \\ u_x &= -\varepsilon_k Q_0(t, u), \quad x = 0; \quad u_x = Q_R(t, u), \quad x = R. \end{aligned} \quad (7)$$

Here $\kappa(u) = \gamma(T(u)) dT(u) / du$, $W(x, t, u) = w(x, t, T(u))$, $Q_0(t, u) = q_0(x, t, T(u))$, $Q_R(t, u) = q_R(x, t, T(u))$.

To derive an approximate solution of problem (7), we make use of the method of lines [11]. By introducing the notation $t_n = n\tau$, $u_n(x) = u(x, t_n)$, $z_n = z(t_n)$, $W_n(x, u_n) = W(x, t_n, u_n)$, $Q_{0n}(u_n) = Q_0(t_n, u_n)$, $Q_{Rn}(u_n) = Q_R(t_n,$

u_n) we obtain the following nonlinear boundary-value problem for evaluation of the function $u_n(x)$ and the constant z_n on the current time layer $t = t_n$:

$$\begin{aligned} \Delta_k u_n &= \kappa(u_n)(u_n - u_{n-1})/(\sigma\tau) - W_n(x, u_n) + \rho x^k \delta_{(k)}(x - z_n)(z_n - z_{n-1})/(\sigma\tau) - \\ &\quad - (1 - \sigma)(\Delta_k u_{n-1} + W_{n-1}(x, u_{n-1}))/\sigma, \quad 0 < x < R; \\ u_{nx} &= -\varepsilon_k Q_{0n}(u_n), \quad x = 0; \quad u_{nx} = Q_{Rn}(u_n), \quad x = R; \quad u_n(z_n) = u_*, \quad n = 1, 2, \dots \end{aligned} \quad (8)$$

Let us show that boundary-value problem (8) approximates Stefan problem (7) for $t = t_n$ with the accuracy $O(\tau)$ for $\sigma = 1$ and $O(\tau^2)$ for $\sigma = 1/2$. It is obvious that this statement holds for $x \neq z_n$ [11]. Let us multiply the differential equation of problem (8) by x^k and integrate over x from $z_n - \varepsilon$ to $z_n + \varepsilon$ ($\varepsilon > 0$). Upon passing to the limit as $\varepsilon \rightarrow 0$, we obtain the relationship

$$\begin{aligned} u_{nx}(z_n + 0) - u_{nx}(z_n - 0) &= p(z_n - z_{n-1})/(\sigma\tau) - \\ &\quad - (1 - \sigma)(u_{n-1,x}(z_n + 0) - u_{n-1,x}(z_n - 0))/\sigma, \end{aligned}$$

which is an approximation at $t = t_n$ of the well-known Stefan condition $u_x(z(t) + 0, t) - u_x(z(t) - 0, t) = p\dot{z}(t)$ with accuracy $O(\tau)$ for $\sigma = 1$ and $O(\tau^2)$ for $\sigma = 1/2$.

Inasmuch as the Green function of the linear boundary-value problem corresponding to Eq. (8) does not exist, then, representing the boundary condition at $x = R$ in the form

$$u_{nx}(R) + hu_n(R) = Q_{Rn}(u_n(R)) + hu_n(R) \quad (0 < h < \infty),$$

we introduce an auxiliary Green function $G(x, y)$ as the solution of the following boundary-value problem:

$$\Delta_k G_k(x, y) = -\delta_{(k)}(x - y), \quad 0 < x < R; \quad G_{kx}(0, y) = 0, \quad G_{kx}(R, y) + hG_k(R, y) = 0,$$

from which we find

$$G_0(x, y) = [1 + h(R - y)]/h, \quad G_1(x, y) = [1 + hR \ln(R/y)]/(hR),$$

$$G_2(x, y) = [hR^2 + (1 - hR)y]/(hR^2y), \quad x \leq y; \quad G_k(y, x) = G_k(x, y), \quad k = 0; 1; 2.$$

It should be noted that to derive the auxiliary Green function, instead of transforming the boundary condition at $x = R$, one can transform the boundary condition at $x = 0$ or both boundary conditions simultaneously. Our choice here was dictated only by the goal of maximum simplification of subsequent solution of the problem, since $u_{nx}(0) = 0$ for $k = 1, 2$.

By making use of the second Green formula for the operator Δ_k as applied to the functions u_n and G_k , we obtain the following integral representation for the solution $u_n(x)$ of boundary-value problem (8):

$$\begin{aligned} u_n(x) &= u_n(R) + Q_{Rn}(u_n(R))/h + \varepsilon_k G_k(x, 0) Q_{0n}(u_n(0)) - \\ &\quad - \rho z_n^k (z_n - z_{n-1}) G_k(x, z_n)/(\sigma\tau) - \int_0^R G_k(x, y) F_{k,n}(y, u_n(y)) y^k dy, \quad 0 \leq x \leq R, \end{aligned} \quad (9)$$

where

$$F_{k,n}(y, u) = \kappa(u)(u - u_{n-1})/(\sigma\tau) - W_n(y, u) - (1 - \sigma)(\Delta_k u_{n-1} + W_{n-1}(y, u_{n-1}))/\sigma.$$

Relationship (9) is a Hammerstein-type integral equation with respect to $u_n(x)$ that contains the unknown constants $u_n(0)$ (for $k = 0$), $u_n(R)$, and z_n . For their evaluation one should write three additional equations by setting $x = 0$, $x = R$, and $x = z_n$, respectively, in (9). We present only the equation derived for $x = R$:

$$R^k [\varepsilon_k Q_{0n}(u_n(0)) + Q_{Rn}(u_n(R))] - \rho z_n^k (z_n - z_{n-1}) / (\sigma \tau) - \int_0^R F_{k,n}(y, u_n(y)) y^k dy = 0.$$

Noteworthy, it does not contain the auxiliary parameter h and it expresses the well-known condition [8] of solvability of stationary problem (8) if it is regarded as the second boundary-value problem for a one-dimensional Poisson equation.

A numerical solution of the system of nonlinear integral equations obtained can be conveniently sought using the projection-grid zonal method [7], which makes it possible to reduce the system to a system of $(N+2+\varepsilon_k)$ nonlinear algebraic equations with respect to $u_n(0)$ (at $k = 0$), $u_n(R)$, z_n , and values of the function $u_n(x)$ averaged over the subintervals (x_{i-1}, x_i) of the segment $[0, R]$:

$$u_{ni} = (k+1) \int_{x_{i-1}}^{x_i} u_n(x) x^k dx / (x_i^{k+1} - x_{i-1}^{k+1}), \quad i = \overline{1, N}.$$

In the averaging, one should take into account the dependence of the functions $G_k(x, z_n)$ and $F_{k,n}(x, u_n(x))$ on the unknown value z_n ($F_{k,n}(x, u_n(x))$ is discontinuous when x passes through the z_n value). Therefore, at first glance, it becomes necessary to construct a subdivision of the segment $[0, R]$ that would somehow be related to the sought value z_n , which should require adjustment of the grid in each time step, as is done in many numerical methods of solving the Stefan problem [3-5]. However, this difficulty can be overcome using an efficient iterative method, e.g., Newton's method, by applying it to the system of nonlinear algebraic equations as a whole. As a result, in each iteration a system of linear algebraic equations will be obtained whose coefficients depend on the known approximation of the solution in the previous iteration. Here a stable subdivision of the segment $[0, R]$ will be retained. The solution of the system in the preceding time layer is appropriate for use as an initial approximation for Newton's method. Then convergence of the iterative process will be achieved by decreasing the step τ .

Now we dwell on distinctive features of solving one-dimensional Stefan problems with a movable external boundary. Here cases are possible where the external boundary coincides with the phase front or simply moves but is not the phase interface. The simplest example of problems of the first type is the single-phase Stefan problems ($k = 0, 1, 2$)

$$L_k T = -w(x, t, T), \quad 0 < x < z(t), \quad t > 0;$$

$$T(x, 0) = T^0(x); \quad T^0(z_0) = T_*; \quad T(z(t), t) = T_*; \quad z(0) = z_0, \quad z_0 \geq 0; \quad (10)$$

$$\lambda(T) T_x = -\varepsilon_k q_0(t, T), \quad x = 0; \quad \lambda(T_*) T_x = q_z(t, T_*) - \rho \dot{z}(t), \quad x = z(t).$$

Problems of the second type arise in mathematical simulation of processes of cryodestruction of biological tissues [12, 13] ($k = 0, 1, 2$):

$$L_k T = -w(x, t, T) - \rho \dot{z}(t) x^k \delta_{(k)}(x - z(t)), \quad x_0 < x < X(t), \quad t > 0;$$

$$T(x, 0) = T^0(x); \quad T^0(z(0)) = T_*; \quad T^0(X(0)) = T_1; \quad T(z(t), t) = T_*; \quad (11)$$

$$\lambda(T) T_x = -q_0(t, T), \quad x = x_0; \quad T_x = 0, \quad x = X(t); \quad T(X(t), t) = T_1.$$

The method of the auxiliary Green function can be used to solve problems (10) and (11). Then, in the case of the single-phase problem (10) we arrive at the integral equation

TABLE 1. Comparison of Computed Values of $z(t)$ with Results Obtained by a Direct-Computation Method [1] and with the Exact Solution

t	$z(t)$				exact value
	method of [1], $\tau = 0.5; \delta = 0.13$	$\tau = 0.5;$ $\delta = 0.1$	$\tau = 0.5;$ $\delta = 0.05$	$\tau = 0.25;$ $\delta = 0.1$	
10	1.469	1.468	1.469	1.469	1.470
20	1.324	1.324	1.324	1.325	1.327
30	1.158	1.161	1.162	1.163	1.166
40	0.968	0.970	0.971	0.975	0.980
50	0.724	0.727	0.726	0.738	0.748

TABLE 2. Comparison of Computed Values of $T(x, t)$ for $\tau = 0.5$ and $\delta = 0.1$ with the Exact Solution

x	$T(x, t)$ at different t					
	10		30		50	
	I	II	I	II	I	II
0	0.9986	1.0000	0.9917	1.0000	0.9447	1.0000
0.2	0.9801	0.9815	0.9623	0.9706	0.8732	0.9286
0.4	0.9245	0.9259	0.8740	0.8824	0.6586	0.7143
0.6	0.8319	0.8333	0.7268	0.7353	0.3009	0.3571
0.8	0.7022	0.7037	0.5208	0.5294	-0.3045	-0.2171
1.0	0.5355	0.5370	0.2559	0.2647	-1.2851	-1.1943
1.2	0.3317	0.3333	-0.1030	-0.0894	-2.4829	-2.3886
1.4	0.0909	0.0926	-0.6841	-0.6706	-3.8979	-3.8000
1.6	-0.2841	-0.2815	-1.3547	-1.3412	-5.5296	-5.4286
1.8	-0.7625	-0.7600	-2.1148	-2.1012	-7.3778	-7.2743
2.0	-1.2974	-1.2948	-2.9643	-2.9506	-9.4420	-9.3371

$$\begin{aligned}
 u_n(x) = & u_* + Q_{zn}(u_*)/h + \varepsilon_k G_{kn}(x, 0) Q_{0n}(u_n(0)) - \\
 & - p(z_n - z_{n-1})/(h\sigma\tau) - \int_0^{z_n} G_{kn}(x, y) F_{k,n}(y, u_n(y)) y^k dy, \quad 0 \leq x \leq z_n,
 \end{aligned} \tag{12}$$

and the integral relationship for the cryodestruction problem (11) is as follows:

$$\begin{aligned}
 u_n(x) = & u_1 + G_{kn}(x, x_0) Q_{0n}(u_n(x_0)) + pz_n^k(z_n - z_{n-1}) G_{kn}(x, z_n)/(\sigma\tau) - \\
 & - \int_0^{x_n} G_{kn}(x, y) F_{k,n}(y, u_n(y)) y^k dy, \quad x_0 \leq x \leq X_n.
 \end{aligned} \tag{13}$$

Expressions for Green functions $G_{kn}(x, y)$ are easily obtained from the above-presented expressions for $G_k(x, y)$ by substituting z_n and X_n , respectively, for R in them. Relationship (12) should be supplemented with integral equations derived from it for $x = 0$ (for $k = 0$) and $x = z_n$, and relationship (13) should be supplemented with three equations obtained for $x = x_0$, $x = z_n$, and $x = X_n$. To solve the obtained systems of integral equations numerically using the zonal method, one can construct a uniform grid by setting the value of its step δ . The coordinate of the last node of the grid in each time layer will coincide with the sought value z_n (or X_n , respectively), which will be found in the course of solving the system of nonlinear algebraic equations under consideration by an iterative method. Here the distance between the penultimate and last nodes of the grid will be less than or equal to the step δ , and the total number of nodes (and therefore the number of equations of the algebraic system) can increase not only upon passing to the next time layer but also upon passing to the next iteration in the given time layer.

The performance of the method for numerical solution of time-dependent Stefan problems proposed in the present paper was tested by comparing results of computations with the numerical solution obtained in [1] using a direct-computation method and with the exact solution $z(t) = a(t_0 - t)^{1/2}$, $0 \leq t < t_0$; $T(x, t) = B_L - A_L x^2 / (t_0 - t)$, $0 \leq x \leq z(t)$; $T(x, t) = B_S - A_S x^2 / (t_0 - t)$, $z(t) \leq x \leq R$; $A_S = (4\lambda_L(B_L - T_*) + pa^2) / (4\lambda_S a^2)$; $A_L = (B_L - T_*) / a^2$; $B_S = T_* + a^2 A_S$ of a problem with cylindrical symmetry taken from [1], which we wrote in the form (6) (for $k = 1$, $\dot{z}(t) < 0$).

Calculations were carried out for $t_0 = 64$, $T_* = 0$, $B_L = 1$, $a = 0.2$, $p = 1$, $\lambda_S = 0.5$, $\lambda_L = 0.75$, $\gamma_S = 2$, $\gamma_L = 1.25$, $R = 2$, $\sigma = 1$, the time-step values $\tau = 0.5$ and $\tau = 0.25$, and the spatial-step values $\delta = 0.1$ ($N = 20$) and $\delta = 0.05$ ($N = 40$). The results do not depend on the value of the auxiliary parameter h ($h \neq 0$).

Tables 1 and 2 present results of computations of values of $z(t)$ and $T(x, t)$ obtained after three iterations of Newton's method (Samarskii and Moiseenko [1] also present results obtained after three iterations). The iteration process converges rapidly: after three iterations an accuracy of 0.001 is reached (as evaluated from solutions of the system of algebraic equations for the preceding and following iterations). A decrease in the time step yields better results than a similar decrease in the step in the spatial variable.

Thus, the numerical-analytical method proposed in the present work makes it possible to find approximate solutions of multidimensional stationary and one-dimensional time-dependent Stefan problems. Application of this method to multidimensional time-dependent problems requires separate consideration. In the multidimensional case, the class of problems that can be treated by the method is determined by the possibility of deriving the Green function for the specific shapes of the regions and the bounding surfaces. The method proposed is characterized by logical simplicity and is as good in efficiency as the direct-computation scheme [1].

NOTATION

T , temperature; x , spatial coordinate; t , time; T_* , phase-transition temperature; $T^0(x)$, initial temperature distribution; λ_S and λ_L , thermal conductivity of the solid and liquid phases, respectively; $\gamma = cp$; c , specific heat; ρ , density; w , distribution of intrinsic heat sources; $z(t)$, coordinate of the phase transition; $\dot{z}(t) = dz/dt$; $p = L\rho_L(T_*)$; L , latent heat of the phase transition; ρ_L , density of the liquid phase; q_0, q_R, q_z , heat fluxes on the boundaries $x = 0$, $x = R$, $x = z(t)$, respectively; $\delta_{(k)}(x)$, Dirac's delta-function with the weight x^k ; $u(T)$, Kirchhoff transform applied to T ; $u_* = u(T_*)$; $\Delta u = \text{div}(\text{grad } u)$, Laplace operator; $\Delta_k u = x^{-k} \partial(x^k u_x) / \partial x$, Laplace operator in the one-dimensional case; $T_x = \partial T / \partial x$; $T_t = \partial T / \partial t$; $u_{nx} = \partial u_n / \partial x$; T_1 , normal temperature of biotissue; $u_1 = u(T_1)$; x_0 , biotissue surface coordinate; $X(t)$, cryoaction isotherm coordinate.

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